Thursday 12 June 2014 – Afternoon

A2 GCE MATHEMATICS

4727/01 Further Pure Mathematics 3

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4727/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

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1 **(i)** Find a vector equation of the line of intersection of the planes $2x + y - z = 4$ and $3x + 5y + 2z = 13$. **[4]**

(ii) Find the exact distance of the point (2, 5, –2) from the plane $2x + y - z = 4$. **[2]**

2 Use the substitution $u = y^2$ to find the general solution of the differential equation

$$\frac{dy}{dx} - 2y = \frac{e^x}{y}$$

for y in terms of x. **[8]**

3 **(i)** Solve the equation $z^6 = 1$, giving your answers in the form $re^{i\theta}$, and sketch an Argand diagram showing the positions of the roots. **[4]**

(ii) Show that $(1 + i)^6 = -8i$. **[3]**

(iii) Hence, or otherwise, solve the equation $z^6 + 8i = 0$, giving your answers in the form $re^{i\theta}$. **[3]**

4 The group G consists of the set {1, 3, 7, 9, 11, 13, 17, 19} combined under multiplication modulo 20.

(i) Find the inverse of each element. **[3]**

(ii) Show that G is not cyclic. **[3]**

(iii) Find two isomorphic subgroups of order 4 and state an isomorphism between them. **[5]**

5 Solve the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-x}$$

subject to the conditions $y = \frac{dy}{dx} = 0$ when $x = 0$. **[10]**

6 The line l has equations $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-7}{5}$. The plane Π has equation $4x - y - z = 8$.

(i) Show that l is parallel to Π but does not lie in Π. **[3]**

(ii) The point A (1, –2, 7) is on l. Write down a vector equation of the line through A which is perpendicular to Π. Hence find the position vector of the point on Π which is closest to A. **[4]**

(iii) Hence write down a vector equation of the line in Π which is parallel to l and closest to it. **[1]**

7 **(i)** By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^5\theta \equiv \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$ **[4]**

(ii) Hence solve the equation

$$\sin 5\theta + 4\sin\theta = 5\sin 3\theta$$

for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. **[4]**

8 G consists of the set of matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where a and b are real and $a^2 + b^2 \neq 0$, combined under the operation of matrix multiplication.

(i) Prove that G is a group. You may assume that matrix multiplication is associative. **[6]**

(ii) Determine whether G is commutative. **[2]**

(iii) Find the order of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. **[3]**

END OF QUESTION PAPER

Question			Answer	Marks	Guidance	
1	(i)		$\begin{pmatrix}2\\1\\-1\end{pmatrix}\times\begin{pmatrix}3\\5\\2\end{pmatrix}=\begin{pmatrix}7\\-7\\7\end{pmatrix}=7\begin{pmatrix}1\\-1\\1\end{pmatrix}$	M1 A1		M1 requires evidence of method for cross product or at least 2 correct values calculated
			(eg) $z=0\Rightarrow 2x+y=4, 3x+5y=13\Rightarrow x=1, y=2$	M1		or any valid point e.g.(0, 3, -1), (3, 0, 2)
			$\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$	A1	oe vector form	Must have full equation including '**r** ='
			Alternative: Find one point	M1		
			Find a second point and vector between points	M1		
			multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$	A1		
			$\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$	A1		
			Alternative: Solve simultaneously	M1	to at least expressions for x,y,z parametrically, or two relationship between 2 variables.	
			Point **and** direction found	M1 A1		
			$\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$	A1		
				[4]		

Question			Answer	Marks	Guidance	
1	(ii)		$\dfrac{\lvert 2\times 2+5--2-4\rvert}{\sqrt{2^2+1^2+1^2}}=\dfrac{7}{\sqrt{6}}$	M1 A1	Condone lack of absolute signs for M1 oe surd form. isw	2.86 with no workings scores M1
			Alternative: find parameter for perpendicular meets plane and use to find distance	M1 **[2]**	For complete method with calculation errors	look for $\lambda = -7/6$
2			$u = y^2 \Rightarrow \dfrac{du}{dx} = 2y\dfrac{dy}{dx}$	M1	Correctly finds	Or $\dfrac{dy}{dx}=\dfrac{1}{2}u^{-\frac{1}{2}}\dfrac{du}{dx}$
			so DE $\Rightarrow 2y\dfrac{dy}{dx}-4y^2=2e^x$	M1	or for complete unsimplified substitution	
			$\Rightarrow \dfrac{du}{dx}-4u=2e^x$	A1		Can be implied by next A1
			$I=\exp\int -4\,dx = e^{-4x}$	A1ft		Must have form $\dfrac{du}{dx}+f(x)u=g(x)$ for this mark and any further marks Can be implied by subsequent work
			$e^{-4x}\dfrac{du}{dx}-4e^{-4x}u=2e^{-3x}$	M1*	Multiples through by IF of form e^{kx}, simplifying RHS	
			$ue^{-4x}=-\frac{2}{3}e^{-3x}(+A)$	*M1dep*	Integrates	
			$u=-\frac{2}{3}e^x+Ae^{4x}$	M1dep *	Rearranges to make u or y^2 the subject	No more than 1 numerical error at this step
			$y=\sqrt{-\frac{2}{3}e^x+Ae^{4x}}$	A1	Cao	ignore use of '±'
			Alternative from 4th mark to 6th mark			
			CF: (u=…) Ae^{4x}	A1		
			PI: $u=ke^x,\ \dfrac{du}{dx}=ke^x$	M1*	PI chosen & differentiated correctly	
			$ke^x-4ke^x=2e^x,\quad k=-\dfrac{2}{3}$	M1 dep* **[8]**	Substitutes and solves	

Question			Answer	Marks	Guidance	
3	(i)		$z^6 = 1 \Rightarrow z = e^{2k\pi i/6}$	M1		
			$k = 0,1,2,3,4,5$	A1	Oe exactly 6 roots	accept roots 1, -1 given as integers.
			Diagram	B1	6 roots in right quadrant,	
				B1	correct angles and moduli	as evidenced by labels, circles, or accurate diagram, or by co-ordinates
				[4]		
3	(ii)		$(1+i)^6 = \left(\sqrt{2}\,e^{\frac{1}{4}\pi i}\right)^6$	M1	Attempts modulus-argument form, getting at least 1 correct	
			$8e^{\frac{6}{4}\pi i}$	M1	for (mod)^6 and arg x 6	
			$= -8i$	A1	**ag**	complete argument including start line
			Alternative:			
			$(1+i)^6 = 1+6i+15i^2+20i^3+15i^4+6i^5+i^6$	M1		
			$= 1+6i-15-20i+15+6i\ -1$	M1	no more than 1 term wrong	**Sc 2** for only lines 2 & 3correct
			$= -8i$	A1	**ag**	
			Alternative: $(1+i)^2 = 2i$	M1		
			$(1+i)^6 = (2i)^3$	M1		
			$= -8i$	A1	**ag**	
				[3]		

Question			Answer	Marks	Guidance	
3	(iii)		$z^6 = -8\mathrm{i} \Rightarrow z = (1+\mathrm{i})\mathrm{e}^{2k\pi \mathrm{i}/6}$	M1		
			$= \sqrt{2}e^{i\frac{\pi}{4}}\mathrm{e}^{2k\pi \mathrm{i}/6}$	M1		
			$\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi(1/4+k/3)},\ k = 0,1,2,3,4,5$	A1	or equivalent k	
			Alternative: $z^6 = 8\mathrm{e}^{\mathrm{i}\pi(\frac{3}{2}+2k)}$	M1		
			$\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi(1/4+k/3)},\ k = 0,1,2,3,4,5$	M1 A1 **[3]**		or equivalent: e.g. $\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi(-1/12+k/3)}$ accept unsimplified modulus

Question			Answer	Marks	Guidance	
4	(i)		<table><tr><td>element</td><td>(1)</td><td>3</td><td>7</td><td>9</td><td>11</td><td>13</td><td>17</td><td>19</td></tr><tr><td>inverse</td><td>(1)</td><td>7</td><td>3</td><td>9</td><td>11</td><td>17</td><td>13</td><td>19</td></tr></table>	B1 B1 B1 **[3]**	2 or more 4 or more all 7 correct	Ignore 1
4	(ii)		(1 has order 1) 9,11,19 have order 2	M1	Correctly identifies order of all elements	Allow one error
			$3^2 = 9 \Rightarrow 3^4 = 1$ so order 4 similarly 7,13,17 order 4	B1	justifies order for at least 1 element of order 4	must show workings towards a^4 for demonstration that these elements are order 4`
			no element of order 8 so not cyclic	A1 **[3]**	www	condone "no generator" in place of "no element or order 8"
4	(iii)			M1	For two sets which both contain "1" and all (4) elements' inverses	
				B1	One subgroup of order 4	
			{1,13, 9, 17} and {1, 3, 9, 7}	A1		
				M1	for correspondence of "their" elements of same order	
			$1 \leftrightarrow 1,\ 9 \leftrightarrow 9,\ 3 \leftrightarrow 13,\ 7 \leftrightarrow 17$	A1 **[5]**	or $3 \leftrightarrow 17,\ 7 \leftrightarrow 13$	

Question			Answer	Marks	Guidance	
5			AE: $\lambda^2+5\lambda+6=0$			
			$\lambda=-2,-3$	B1		
			CF: $Ae^{-2x}+Be^{-3x}$	B1ft		
			PI: $y=ae^{-x}$	B1ft		
			$ae^{-x}-5ae^{-x}+6ae^{-x}=e^{-x}$	M1	Differentiate and substitute	
			$2a=1$			
			$a=\frac{1}{2}$	A1		
			GS: $(y=)\frac{1}{2}e^{-x}+Ae^{-2x}+Be^{-3x}$	A1ft		ft must be of form "ke^{-x} plus a standard CF form" with 2 arbitrary constants
			$x=0, y=0 \Rightarrow \frac{1}{2}+A+B=0$	M1	Use condition on GS	Must have 2 arbitrary constants
			$y'=-\frac{1}{2}e^{-x}-2Ae^{-2x}-3Be^{-3x}$	M1*	Differentiate their GS of form $y=ke^{-x}+Ae^{mx}+Be^{nx}$ where k, m, n are non-zero constants and m, n not 1	
			$x=0, y'=0 \Rightarrow -\frac{1}{2}-2A-3B=0$			
			$A=-1,\ B=\frac{1}{2}$	M1dep*	Use condition and attempt to find A, B	
			$y=\frac{1}{2}e^{-x}-e^{-2x}+\frac{1}{2}e^{-3x}$	A1	www	Must have 'y ='
				[10]		

Question			Answer	Marks	Guidance	
6	(i)		$l \parallel \begin{pmatrix}2\\3\\5\end{pmatrix}$ $\Pi \perp \begin{pmatrix}4\\-1\\-1\end{pmatrix}$ so $\begin{pmatrix}2\\3\\5\end{pmatrix}\bullet\begin{pmatrix}4\\-1\\-1\end{pmatrix} = 0 \Rightarrow l \parallel \Pi$	M1	dot product of correct vectors = 0	
			(1, –2, 7) on l but $4\times1 - -2 - 7 = -1 \neq 8$ so not in Π	M1	substitute point on line into Π and calculate d	
			hence l not in Π	A1	Full argument includes key components	Argument can be about a general point on line
				[3]		
6	(ii)		$(\mathbf{r} =)\begin{pmatrix}1\\-2\\7\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\-1\end{pmatrix}$	B1		
			closest point where meets Π $4(1+4\lambda) - (-2-\lambda) - (7-\lambda) = 8$	M1	parametric form of (x, y, z) substituted into plane	
			$\Rightarrow \lambda = \frac{1}{2}$	A1ft		
			$\Rightarrow \mathbf{r} = \begin{pmatrix}3\\-\frac{5}{2}\\\frac{13}{2}\end{pmatrix}$	A1		
				[4]		
6	(iii)		$\mathbf{r} = \begin{pmatrix}3\\-\frac{5}{2}\\\frac{13}{2}\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\5\end{pmatrix}$	B1ft	oe	must have "**r** ="
				[1]		

Question			Answer	Marks	Guidance	
7	(i)		$2\mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}$ $2\mathrm{i}\sin n\theta = \mathrm{e}^{\mathrm{i}n\theta} - \mathrm{e}^{-\mathrm{i}n\theta}$ $(2\mathrm{i}\sin\theta)^5 = (\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})^5$	B1	any equivalent form	If use z, must define it
			$= \mathrm{e}^{\mathrm{i}5\theta} - 5\mathrm{e}^{\mathrm{i}3\theta} + 10\mathrm{e}^{\mathrm{i}\theta} - 10\mathrm{e}^{-\mathrm{i}\theta} + 5\mathrm{e}^{-\mathrm{i}3\theta} - \mathrm{e}^{-\mathrm{i}5\theta}$	M1*	binomial expansion	can be unsimplified
			$32i\sin^5\theta = (e^{5i\theta} - e^{-5i\theta}) - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta})$ $= 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$	M1dep*	grouping terms	Award **B1 then sc M1A1** for candidates who omit this stage from otherwise complete argument
			$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$	A1 **[4]**	AG	must convince on the $\frac{1}{16}$ and on the elimination of i
7	(ii)		$16\sin^5\theta - 10\sin\theta = \sin 5\theta - 5\sin 3\theta$	M1*	Attempts to eliminate sin5θ and sin3θ	
			$16\sin^5\theta - 6\sin\theta = 0$	A1		Or $16\sin^5\theta = 6\sin\theta$
			$\sin\theta = 0, \pm\sqrt[4]{\frac{3}{8}}$	M1dep*	must have 3 values for sin θ	
			$\theta = 0, \pm 0.899$	A1 **[4]**		

Question		Answer	Marks	Guidance	
8	(i)	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is identity	B1		
		$\begin{pmatrix}a & -b\\b & a\end{pmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix}a & b\\-b & a\end{pmatrix} \in G$	M1 A1	for M1, must at least get matrix in form $\begin{pmatrix}x & -y\\y & x\end{pmatrix}$, or state existence of inverse due to non-singularity	
		$\begin{pmatrix}a & -b\\b & a\end{pmatrix}\begin{pmatrix}c & -d\\d & c\end{pmatrix} = \begin{pmatrix}ac-bd & -bc-ad\\bc+ad & ac-bd\end{pmatrix}$	M1		
		and $(ac-bd)^2+(bc+ad)^2 = a^2c^2+b^2d^2+b^2c^2+a^2d^2$ $=(a^2+b^2)(c^2+d^2)\neq 0$	M1 A1 **[6]**	Must not attempt to prove commutativity in (i)	
8	(ii)	$\begin{pmatrix}c & -d\\d & c\end{pmatrix}\begin{pmatrix}a & -b\\b & a\end{pmatrix} = \begin{pmatrix}ac-bd & -bc-ad\\bc+ad & ac-bd\end{pmatrix}$	M1		must also consider matrices reversed, but may be seen in (i)
		$=\begin{pmatrix}a & -b\\b & a\end{pmatrix}\begin{pmatrix}c & -d\\d & c\end{pmatrix}$ so commutative	A1 **[2]**		
8	(iii)	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}^2 = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	M1	g^2 must be correct	
		$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	M1	allow 1 error in getting g^4	
		order 4	A1 **[3]**		